GROWTH IN PRODUCT REPLACEMENT GRAPHS

ANTON MALYSHEV⋆ AND IGOR PAK⋆

Abstract. We prove the exponential growth of product replacement graphs for a large class of groups. Much of our effort is dedicated to the study of product replacement graphs of Grigorchuk groups, where the problem is most difficult.

Introduction

The *product replacement graphs* $\Gamma_k(G)$ are the graphs on generating k-tuples of a group G, with edges corresponding to multiplications of one generator by another (see below). These graphs play an important role in computational group theory (see e.g. [BL, NP, P1]), and are related to the Andrews–Curtis conjecture in algebraic topology (see e.g. [BKM, BLM, Met]). For infinite groups, proving non-amenability of graphs $\Gamma_k(G)$ is a major open problem, closely related to Kazhdan's property (T) of $\mathrm{Aut}(F_k)$. In this paper we establish a weaker property, the exponential growth of product replacement graphs for a large class of infinite groups.

Let us begin by stating the main conjecture we address in this paper. The motivation behind it is postponed until final remarks (see Subsection 7.1).

Main Conjecture: *Let G be an infinite group generated by d elements. Then the product replacement graphs $\Gamma_k(G)$ have exponential growth, for all $k \geq d+1$.*

Formally speaking, graphs $\Gamma_k(G)$ can be disconnected, in which case we conjecture that *at least one* connected component has exponential growth.

We approach the conjecture by looking at the growth of groups. The conjecture is straightforward for groups of exponential growth (see Proposition 3.1). By Gromov's theorem, groups of polynomial growth are virtually nilpotent. This allows us to prove the conjecture for groups of polynomial growth as well (see Proposition 3.3).

Unfortunately, groups of intermediate growth lack the rigid structure of nilpotent groups, so much that even explicit examples are difficult to find and analyze (see e.g. [dlH1, G3]). Even now, much remains open for the classical *Grigorchuk group* $\mathbb{G}$, the first example of a group of intermediate growth discovered by Grigorchuk (see [G1, G2]).

We present a new combinatorial technique which allows us to establish the conjecture for a large class of Grigorchuk groups $\mathbb{G}_\omega$, the main result of this paper (see Theorem 6.2). This technique will be extended to a large class groups in [M2], to include many known examples of groups of intermediate growth (see Subsection 7.4).

The rest of this paper is structured as follows. We begin with basic definitions of growth of groups and the product replacement graphs (Section 1). In Section 2 we present basic results on the growth and connectivity of graphs $\Gamma_k(G)$; we also present general combinatorial tools for establishing the exponential growth results. In Section 3 we prove the conjecture for groups of exponential and polynomial growth, as well as a stronger result for all virtually solvable groups. In a technical Section 4 we describe general tools and techniques for working with subgroups $G \subset \mathrm{Aut}(\mathbf{T}_2)$ and their product replacement graphs. In the next two sections 5 and 6 we establish the main result. First, we prove the exponential growth of $\Gamma_k(\mathbb{G})$ for $k \geq 5$; in this case the (technical) argument is the most lucid. We then generalize this approach to *all* Grigorchuk groups $\mathbb{G}_\omega$. We conclude with final remarks and open problems (Section 7).

⋆Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: {amalyshev,pak}@math.ucla.edu.

1. Background and definitions

1.1. **Notation.** Let X be a finite set. We write $\# X$ or $|X|$ to denote the size of X.

Let Γ be a directed graph, which may have loops and repeated edges. We define $v \in \Gamma$ to mean that v is a vertex of Γ. Let v, w be vertices of Γ. We write $v \to w$ when there is an edge in Γ from v to w, and $v \rightsquigarrow w$ when there is a path in Γ from v to w. We say Γ is *symmetric* if for every edge $v \to w$ of Γ there is an *inverse edge* $w \to v$. Every graph considered in this paper is a symmetric directed graph, unless otherwise specified. When convenient, we think of a symmetric directed graph as an undirected graph by identifying every edge with its inverse.

Let G be a group, which may be finite or infinite. A *generating n-tuple* of G is an element $(g_1, \ldots, g_n) \in G^n$, such that $G = \langle g_1, \ldots, g_n \rangle$. Let $S = (g_1, \ldots, g_n)$ be such an n-tuple. Consider a left action of G on a set X. The *Schreier graph* $\mathrm{Schr}_S(G, X)$ of this action with respect to S, is the directed graph whose vertices are the elements of X, with edges $x \to g_i x$ and $x \to g_i^{-1} x$ for each $x \in X$, and each $0 \le i \le n$. Note that each vertex in $\mathrm{Schr}_S(G, X)$ has $2n$ edges leaving it, and each edge $v \to w$ in such a graph has an inverse edge $w \to v$. Thus, $\mathrm{Schr}_S(G, X)$ is a $2n$-regular symmetric directed graph.

The *Cayley graph* $\mathrm{Cay}_S(G)$ is the Schreier graph $\mathrm{Schr}_S(G, G)$ with respect to the left action of G on itself by multiplication. Clearly, the Cayley graph $\mathrm{Cay}_S(G)$ is connected. Given $g \in G$, we define $\ell_S(g)$ to be the length of the shortest path from 1 to g in the Cayley graph of G.

When the context makes it clear what the generating n-tuple S is, we drop the subscript, and simply write $\mathrm{Cay}(G)$, $\mathrm{Schr}(G, X)$, and $\ell(g)$. We write $\mathrm{Aut}(G)$ for the group of automorphisms of G. We write $H < G$ when H is a subgroup of G, and $H \lneqq G$ when H is a proper subgroup of G. For an element $g \in G$, denote by $\mathrm{ord}(g)$ the order of g. For $g_1, \ldots, g_n \in G$, denote

$$\overrightarrow{\prod_{i=1\ldots n}} g_i \; = \; g_1 \cdots g_n.$$

1.2. **Growth in graphs.** Let Γ be a symmetric directed graph, and let $v \in \Gamma$. The ball of radius r centered at v, denoted $B_\Gamma(v, r)$, is the set of vertices $w \in \Gamma$ such that there is a path of length at most r between v and w. For example, suppose $\Gamma = \mathrm{Cay}_S(G)$. Then $B_\Gamma(1, r)$ consists of the elements $g \in G$ for which $\ell_S(g) \le r$.

We say Γ has *exponential growth from* v, if there is a constant $\alpha > 1$, such that $|B_\Gamma(v, r)| \ge \alpha^r$ for all r (equivalently, for sufficiently large r). Suppose Γ has exponential growth from w, and there is a path $v \rightsquigarrow w$ in Γ. Then Γ also has exponential growth from v. Thus, if Γ is connected and has exponential growth from some $v \in \Gamma$, it also has exponential growth from any $w \in \Gamma$. In this case, we say that Γ has *exponential growth.*

1.3. **Growth in groups.** Let G be a group, Let S be a generating n-tuple of G. Define $B_{G,S}(r) = B_\Gamma(1, r)$, where $\Gamma = \mathrm{Cay}_S(G)$. When it is clear what S is, we simply write $B_G(r)$ instead. It is easy to verify that the following definitions are independent of the choice of generators S.

We say G has *exponential growth* if Γ has exponential growth. In other words, G has exponential growth if there is a constant $\alpha > 1$ such that $|B_G(r)| \ge \alpha^r$ for sufficiently large r. Equivalently G has exponential growth if and only if

$$\liminf_{r \to \infty} \frac{\log \big|B_G(r)\big|}{r} > 0.$$

Similarly, we say G has *polynomial growth* if there is a constant d with $|B_G(r)| \le r^d$ for sufficiently large r. In other words, G has polynomial growth if

$$\limsup_{r \to \infty} \frac{\log \big|B_G(r)\big|}{\log r} < \infty.$$

Example 1.1. The group $\mathbb{Z}$ has polynomial growth. With respect to the generating 1-tuple $S = (1)$, we have $B_{\mathbb{Z}}(r) = [-r, r]$, and hence $|B_{\mathbb{Z}}(r)| = 2r + 1$.

Example 1.2. The free group with two generators, $G = F_2 = \langle a, b\rangle$ has exponential growth. With respect to the generators $S = (a, b)$, we have $|B_G(r)| = 1 + 4 \cdot 3^{r-1}$ for $r \geq 1$.

We say G has *intermediate growth* if it has neither exponential nor polynomial growth. The first known example of a group of intermediate growth is the Grigorchuk group $\mathbb{G}$, which will be defined later, in Section 5. We refer to [dlH1, §VI] and [GP] for more on the growth of groups (see also §7.4).

1.4. **Product replacement graphs.** Given a generating n-tuple of S a group G, we can take an element of S and multiply it, either on the left or the right, by another element or another element's inverse. Such an operation is called a *Nielsen move*. Formally, for each $1 \leq i, j \leq n$ with $i \neq j$, we define the Nielsen moves $R_{ij}^{\pm 1}$, $L_{ij}^{\pm 1}$ by

$$R_{ij}^{\pm 1}(g_1, \ldots, g_i, \ldots, g_j, \ldots g_n) = (g_1, \ldots, g_i, \ldots, g_j g_i^{\pm 1}, \ldots, g_n),$$
$$\text{and} \quad L_{ij}^{\pm 1}(g_1, \ldots, g_i, \ldots, g_j, \ldots g_n) = (g_1, \ldots, g_i, \ldots, g_i^{\pm 1} g_j, \ldots, g_n).$$

Clearly, if S is a generating n-tuple of G, then $R_{ij}S$, $R_{ij}^{-1}S$, $L_{ij}S$, and $L_{ij}^{-1}S$ are also generating n-tuples of G.

We define the *product replacement graph* $\Gamma_n(G)$ to be the directed graph whose vertices are the generating n-tuples of G, where there is an edge from S to $R_{ij}S$, $R_{ij}^{-1}S$, $L_{ij}S$, and $L_{ij}^{-1}S$, for each generating n-tuple S and each pair of integers $i \neq j$ satisfying $1 \leq i, j \leq n$. This is a $4n(n-1)$-regular symmetric directed graph.

Observe that

$$\begin{aligned} R_{ij} L_{ji}^{-1} L_{ij}(g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) \;&=\; R_{ij} L_{ji}^{-1}(g_1, \ldots, g_i, \ldots, g_i g_j, \ldots, g_n) \\ =\; R_{ij}(g_1, \ldots, g_j^{-1}, \ldots, g_i g_j, \ldots, g_n) \;&=\; (g_1, \ldots, g_j^{-1}, \ldots, g_i, \ldots, g_n). \end{aligned}$$

Hence, a series of Nielsen moves can swap two elements in a generating n-tuple, inverting one of them. Doing this twice simply inverts both elements. This implies that Nielsen moves permit us to rearrange generators in an n-tuple, except that we may need to invert one element (see [P1]). Moreover, if $g_i = 1$ for some i, then we can use Nielsen moves invert any one element, and therefore we can rearrange the generators freely.

Example 1.3. The graph $\Gamma_2(\mathbb{Z})$ has a vertex for each pair of relatively prime integers (a, b), with two edges from (a, b) to each of $(a, b + a)$, $(a, b - a)$, $(a + b, b)$ and $(a - b, b)$. It is easy to check that this graph has exponential growth (see Lemma 2.9).

Example 1.4. The graph $\Gamma_2(\mathbb{Z}^2)$ is disconnected. It has two connected components, each isomorphic to a Cayley graph of $SL_2(\mathbb{Z})$.

Example 1.5. Let $G = \mathbb{Z}_p^n$, with p prime. Then $\Gamma_n(G)$ is the set of bases of $\mathbb{Z}_p^n$ as a vector space over $\mathbb{Z}_p$. These bases are in one-to-one correspondence with matrices in $GL_n(\mathbb{Z}_p)$, and Nielsen moves correspond to elementary row operations. Row operations do not change the determinant of a matrix. It follows that there is one connected component for every value of the determinant. This implies that $\Gamma_n(\mathbb{Z}_p^n)$ has $p - 1$ connected components (see [DG]).

1.5. **Growth of $\Gamma_n(G)$.** Let $S = (g_1, \ldots, g_n) \in \Gamma_n(G)$. We write

$$S^{(m)} := (g_1, \ldots, g_n, 1, \ldots, 1) \in \Gamma_{n+m}(G),$$

and define $\Gamma_{n+m}(G, S)$ to be the connected component of $\Gamma_{n+m}(G)$ containing $S^{(m)}$.

We say G has *exponential Nielsen growth* if $\Gamma_n(G, S)$ has exponential growth for some n and some generating n-tuple S of G. It is easy to show (see section 3) that a finitely generated group G has exponential Nielsen growth if G is either an infinite group of polynomial growth, or a group of exponential growth. This suggests that every infinite finitely generated group has exponential Nielsen growth:

Conjecture 1.6. *For every infinite finitely generated group G, there is an generating n-tuple $S \in \Gamma_n(G)$ such that $\Gamma_n(G, S)$ has exponential growth.*

This is a variation of the Main Conjecture in the introduction.

2. Basic results

2.1. **Growth of graphs.** We do not need to prove that $B_\Gamma(v,r)$ is large for every single r to conclude that Γ has exponential growth from v. As the following lemma shows, it suffices to prove it for a relatively sparse set of numbers r.

A sequence of positive integers $r_1, r_2, \ldots$ is called *log-dense* if it is increasing, and there is a constant β such that $r_{i+1} \le \beta r_i$ for every $i \ge 1$. In other words, an increasing integer sequence (r_i) is log-dense if the gaps in the sequence $(\log r_i)$ are bounded above.

Lemma 2.1. *Let Γ be a symmetric directed graph, and let v be a vertex of Γ. Suppose that for some constant $\alpha > 1$, there is a log-dense sequence $r_1, r_2, \ldots$ such that $|B(v, r_i)| \ge \alpha^{r_i}$ for every $i \ge 1$. Then Γ has exponential growth from v.*

Proof. Since r_i is an increasing sequence of positive integers, we can conclude that for sufficiently large r, there is an i with $r_i \le r \le r_{i+1}$. Since $r_{i+1} \le \beta r_i$, we have $r_i \ge r/\beta$. Thus,

$$|B(v,r)| \ge |B(v,r_i)| \ge \alpha^{r_i} \ge \alpha^{r/\beta},$$

which implies the result. □

If a graph Γ is a covering of another graph Γ', and Γ' has exponential growth, then so does Γ.

Proposition 2.2. *Let Γ' and Γ be symmetric directed graphs, and suppose $\phi : \Gamma' \to \Gamma$ maps the set of neighbors of each vertex $v \in \Gamma'$ surjectively onto the neighbors of $\phi(v)$. Suppose Γ has exponential growth from $\phi(w)$. Then Γ' has exponential growth from w.*

Proof. It suffices to show that ϕ maps $B_{\Gamma'}(w,r)$ onto $B_\Gamma(\phi(w),r)$ for all $r \ge 0$, since in that case

$$|B_{\Gamma'}(w,r)| \ge |B_\Gamma(\phi(w),r)|\,.$$

We prove this by induction on r. The base case $r = 0$ is trivial. Suppose

$$\phi\bigl(B_{\Gamma'}(w,r)\bigr) \supseteq B_\Gamma\bigl(\phi(w),r\bigr),$$

and consider $v \in B_\Gamma(\phi(w), r+1)$. We know that v has a neighbor $u \in B_\Gamma(\phi(w),r)$, which has a preimage $u' \in B_{\Gamma'}(w,r)$. Since v is a neighbor of u, we know that some neighbor of u' is mapped to v. Therefore, $v \in \phi\bigl(B_{\Gamma'}(w,r+1)\bigr)$, as desired. □

It is easy to see that if a graph Γ is a subgraph of Γ', and Γ has exponential growth, so does Γ'. Moreover, we have the following stronger result:

Proposition 2.3. *Let Γ and Γ' be symmetric directed graphs, and suppose $\phi : \Gamma \to \Gamma'$ sends neighbors to neighbors. Suppose that there is a constant C such that $\#\,\phi^{-1}(v') \le C$ for every vertex $v' \in \Gamma'$. Suppose that Γ has exponential growth from w. Then Γ' has exponential growth from $\phi(w)$.*

Proof. It suffices to show that ϕ maps $B_\Gamma(w,r)$ into $B_{\Gamma'}(\phi(w),r)$ for all $r \ge 0$, since in that case

$$|B_{\Gamma'}(\phi(w),r)| \ge |B_\Gamma(w,r)|\,/C.$$

We prove this by induction or r. The base case $r = 0$ is trivial. Suppose

$$\phi\bigl(B_\Gamma(w,r)\bigr) \subseteq B_{\Gamma'}\bigl(\phi(w),r\bigr),$$

and consider $v \in B_\Gamma(w,r+1)$. We know that v has a neighbor $u \in B_\Gamma(w,r)$, and $\phi(u) \in B_{\Gamma'}(\phi(w),r)$. Since u and v are neighbors, and ϕ sends neighbors to neighbors, we see that $\phi(v)$ is a neighbor of $\phi(u)$. It follows that $\phi(v) \in B_{\Gamma'}(\phi(w), r+1)$, as desired. □

2.2. **Growth of product replacement graphs.** Observe that if $m \geq n$ then $\Gamma_n(G,S)$ embeds into $\Gamma_m(G,S)$. Therefore, by Lemma 2.3 if $\Gamma_n(G,S)$ has exponential growth, so does $\Gamma_m(G,S)$.

Moreover, if H is a finitely generated subgroup of G, then every product replacement graph of H embeds in some product replacement graph of G. We can conclude that if a subgroup of G has a product replacement graph of exponential growth, so does G. Formally:

Proposition 2.4. *Let H and G be finitely generated groups with $H < G$. Suppose some connected component of $\Gamma_m(H)$ has exponential growth, and let $S \in \Gamma_n(G)$. Then $\Gamma_{n+m}(G,S)$ has exponential growth. In particular, if $H < G$ and H has exponential Nielsen growth, then G also has exponential Nielsen growth.*

Proof. Let $S = (g_1, \dots, g_n) \in \Gamma_n(G)$. We know that $\Gamma_m(H)$ has exponential growth from some $T \in \Gamma_m(H)$. Let $T = (h_1, \dots, h_m)$. There is a graph embedding $\phi : \Gamma_m(H) \to \Gamma_{n+m}(G)$ given by

$$\phi(h'_1, \dots, h'_m) = (g_1, \dots, g_n, h'_1, \dots, h'_m).$$

Hence, $\Gamma_{n+m}(G)$ has exponential growth from $\phi(T)$. Since the g_i's generate G, we know that each h_i is a product of g_i's and their inverses. Thus, there is a sequence of Nielsen moves $S^{(m)} \rightsquigarrow \phi(T)$, where

$$S^{(m)} = (g_1, \dots, g_n, 1, \dots, 1), \text{ and } \phi(T) = (g_1, \dots, g_n, h_1, \dots, h_m).$$

Therefore, $\Gamma_{n+m}(G,S) = \Gamma_{n+m}\big(G, \phi(T)\big)$, which implies that $\Gamma_{n+m}(G,S)$ has exponential growth. □

Similarly, we can show that if a group quotient of G has a product replacement graph of exponential growth, then so does G.

Proposition 2.5. *Let G and H be finitely generated groups, and let $f : G \to H$ be a surjective group homomorphism. Let $S \in \Gamma_n(G)$. Then the following hold.*

(1) *Suppose $\Gamma_n\big(H, f(S)\big)$ has exponential growth. Then $\Gamma_n(G,S)$ has exponential growth.*
(2) *Suppose some connected component of $\Gamma_m(H)$ has exponential growth. Then $\Gamma_{n+m}(G,S)$ has exponential growth.*
(3) *Suppose H has exponential Nielsen growth. Then G also has exponential Nielsen growth.*

Proof. For (1), we extend f to a map $\Gamma_n(G) \to \Gamma_n(H)$ by making the following definition.

$$f(g_1, \dots, g_n) = \big(f(g_1), \dots, f(g_h)\big).$$

This map f sends the neighbors of every $T \in \Gamma_n(G)$ surjectively onto the neighbors of $f(T)$. Thus, since $\Gamma_n(H)$ has exponential growth from $f(S)$, we can apply Proposition 2.2, and conclude that $\Gamma_n(G)$ has exponential growth from S.

For (2), let $S = (g_1, \dots, g_n) \in \Gamma_n(G)$, and choose

$$T = (h_1, \dots, h_m) = \big(f(\tilde{h}_1), \dots, f(\tilde{h}_m)\big) \in \Gamma_m(H)$$

such that $\Gamma_m(H,T)$ has exponential growth. Then

$$\Gamma_{n+m}\big(H, (f(g_1), \dots, f(g_n), h_1, \dots, h_m)\big)$$

also has exponential growth. Thus, by (1),

$$\Gamma_{n+m}\big(G, (g_1, \dots, g_n, \tilde{h}_1, \dots, \tilde{h}_m)\big)$$

has exponential growth. Since the g_i's generate G, we know that there is a path in $\Gamma_{n+m}(G)$

$$(g_1, \dots, g_n, \tilde{h}_1, \dots, \tilde{h}_m) \rightsquigarrow (g_1, \dots, g_n, 1, \dots, 1) = S^{(m)}.$$

Hence, $\Gamma_{n+m}(G)$ also has exponential growth from $S^{(m)}$, i.e. $\Gamma_{n+m}(G,S)$ has exponential growth. Finally, part (3) follows immediately from (2). □

In a different direction, if G has a product replacement graph of exponential growth, so does every quotient of H by a finite subgroup.

Proposition 2.6. *Let G and H be finitely generated groups, and let $f : G \to H$ be a surjective group homomorphism with finite kernel. For every $S \in \Gamma_n(G)$, if $\Gamma_n(G, S)$ has exponential growth, then $\Gamma_n\big(H, f(S)\big)$ has exponential growth. In particular, if G has exponential Nielsen growth, then H also has exponential Nielsen growth.*

Proof. We extend the map $f : G \to H$, to the map $f : \Gamma_n(G) \to \Gamma_n(H)$, given by

$$f(g_1, \ldots, g_n) = \big(f(g_1), \ldots, f(g_h)\big).$$

This map sends neighbors to neighbors, and the preimage of each vertex has bounded size. The graph $\Gamma_n(G)$ has exponential growth from S. Hence, by Proposition 2.3, $\Gamma_n(H)$ has exponential growth from $f(S)$. □

We summarize the previous three results in the following proposition.

Proposition 2.7. *Let G and G' be finitely generated groups, and suppose G is subgroup, quotient, or extension by a finite group of G'. If G has exponential Nielsen growth, then G' also has exponential Nielsen growth.*

This gives us an easy way to prove that a fairly large class of groups have exponential Nielsen growth.

Lemma 2.8. *Let G be a finitely generated group. Suppose G contains an element of infinite order. For every $S \in \Gamma_n(G)$ and every $m \geq n + 2$, we have that $\Gamma_m(G, S)$ has exponential growth.*

Proof. By assumption, the group G contains a subgroup isomorphic to $\mathbb{Z}$. It is easy to see that $\Gamma_2(\mathbb{Z})$ has exponential growth (see e.g. Lemma 2.9). By Proposition 2.4, it follows that $\Gamma_{n+2}(G, S)$ has exponential growth, and hence so does $\Gamma_m(G, S)$ for every $m \geq n + 2$. □

The proof above uses the following elementary result. We include a short proof for completeness.

Lemma 2.9. *The product replacement graph $\Gamma_2(\mathbb{Z})$ has exponential growth.*

Proof. Think of $\Gamma_2(\mathbb{Z})$ as an undirected graph by identifying each edge with its inverse edge. Consider the induced subgraph Γ' of $\Gamma_2(\mathbb{Z})$ consisting of those (a, b) for which $a, b > 0$. Choose directions for the edges so that (a, b) points to (c, d) if $a + b < c + d$. (The following enumeration shows that $a+b = c+d$ does not occur.) If $a = b$, then $(a, b) = (1, 1)$ which has two children in Γ': $(1, 1) \to (1, 2)$ and $(1, 1) \to (2, 1)$. If $a > b$, then (a, b) has exactly three neighbors in Γ', two outgoing and one incoming: $(a - b, b) \to (a, b)$, $(a, b) \to (a + b, b)$, and $(a, b) \to (a, a + b)$. The same is true if $a < b$. Thus, Γ' contains an infinite binary tree rooted at $(1, 1)$, which has exponential growth, and hence so does $\Gamma_2(\mathbb{Z})$. □

2.3. **Effective results.** Many infinite groups, including the Grigorchuk group, do not contain an element of infinite order. This means we need a different approach to proving that such groups have exponential Nielsen growth. One strategy is to make Lemma 2.8 effective: even if a group G has no element of infinite order, if the orders some elements G grow exponentially with their length, then G has a product replacement graph with exponential growth.

Proposition 2.10. *Let G be a finitely generated group, and fix a generating n-tuple $S \in \Gamma_n(G)$. Let $\alpha > 1$ be a constant, and let (r_i) be a log-dense sequence. Suppose there is a sequence (t_i) in G, such that $\ell(t_i) \leq r_i$, and $\operatorname{ord}(t_i) \geq \alpha^{r_i}$. Then $\Gamma_m(G, S)$ has exponential growth for every $m \geq n + 2$.*

Remark 2.11. Observe that not every infinite group satisfies the hypotheses of Proposition 2.10. For instance, the Burnside groups $B(m, n)$ with $m \geq 2$ and odd $n \geq 665$ are infinite, finitely generated with m generators, and satisfy $x^n = 1$ for each $x \in B(m, n)$ (see [Adi]). In fact, the proposition does not even apply to the Grigorchuk group $\mathbb{G}$, since all its elements have orders which are polynomial in their length (see Proposition 5.4).

Our proof of Proposition 2.10 relies on an effective version of Lemma 2.9. Since $\mathbb{Z}_n$ is finite, we cannot have indefinite exponential growth of balls in $\Gamma_2(\mathbb{Z}_n)$. However, $SL_2(\mathbb{Z})$ has property (τ) with respect to its congruence subgroups [Lub], which implies that $\Gamma_2(\mathbb{Z}_n)$ is an expander graph. Thus, its balls grow exponentially until they cover half the graph. For our purposes, a weaker result with a more elementary proof suffices.

Lemma 2.12. *There are universal constants $\nu > 1$ and $d > 0$, such that for sufficiently large n and r we have*

$$\left|B_{\Gamma_2(\mathbb{Z}_n)}(S,r)\right| \geq \min(\nu^r, n^d),$$

where S is the generating 2-tuple $(1,0) \in \Gamma_2(\mathbb{Z}_n)$.

Proof. We make the definitions

$$\Gamma = \Gamma_2(\mathbb{Z}), \quad \Gamma' = \Gamma_2(\mathbb{Z}_n), \quad S = (1,0) \in \Gamma, \quad \text{and} \quad S' = (1,0) \in \Gamma'.$$

For $(a,b) \in \Gamma$, define $|(a,b)| = \max\{|a|, |b|\}$. Note that a Nielsen move at most doubles $|(a,b)|$. Hence, for any $(a,b) \in B_\Gamma(S,r)$, we have $|(a,b)| \leq 2^r$. Thus, if $n > 2^{r+1}$, the natural projection from Γ to Γ' is injective on the ball $B_\Gamma(S,r)$. It follows that $|B_{\Gamma'}(S',r)| = |B_\Gamma(S,r)|$.

By Lemma 2.9, there is a constant $\nu > 1$ with $|B_\Gamma(S,r)| \geq \nu^r$ for all $r \geq 0$. Thus, as long as $r < \log_2(n) - 1$, we have $|B_{\Gamma'}(S',r)| \geq \nu^r$. Define r_0 to be the greatest integer less than $\log_2(n) - 1$, i.e. $r_0 = \lceil \log_2(n) \rceil - 2$. Then, for every $r \geq \log_2(n) - 1$, we have $r > r_0$, so

$$|B_{\Gamma'}(S',r)| \geq |B_{\Gamma'}(S',r_0)| \geq \nu^{\lceil \log_2(n) \rceil - 2} \geq n^{\log_2(\nu)}/\nu^2 \geq n^d$$

for sufficiently large n, and $0 < d < \log_2(\nu)$. This completes the proof. □

Proof of Proposition 2.10. It is enough to show that $\Gamma_{n+2}(G,S)$ has exponential growth. Denote $\Gamma = \Gamma_{n+2}(G)$, and $M_i = \operatorname{ord}(t_i)$. Let $S = (g_1, \dots, g_n) \in \Gamma_n(G,S)$. Since $\ell(t_i) \leq r_i$, there exists a path of at most r_i Nielsen moves from $S^{(2)} = (g_1, \dots, g_n, 1, 1)$ to $(g_1, \dots, g_n, t_i, 1)$. The last two generators in that $(n+2)$-tuple generate a cyclic group of order M_i. Applying Lemma 2.12 to that part of the $(n+2)$-tuple guarantees that for sufficiently large r_i, we can reach at least $\min(\nu^{r_i}, M_i^d)$ different $(n+2)$-tuples within r_i more Nielsen moves. Thus, if r_i is sufficiently large,

$$\left|B_\Gamma(S^{(2)}, 2r_i)\right| \geq \min(\nu^{r_i}, M_i^d) \geq \min(\nu^{r_i}, \alpha^{dr_i}) \geq \min(\nu, \alpha^d)^{r_i}$$

Thus, by Lemma 2.1, the graph Γ has exponential growth from $S^{(2)}$, which completes the proof. □

Since Proposition 2.10 is not strong enough to prove that the Grigorchuk group has exponential Nielsen growth, we use a different approach. Instead of using large cyclic groups in G, we use large cubes in G, as follows.

Let G be any group, and let $(g_1, \dots, g_k) \in G^k$, we say the *cube spanned by* $(g_1, \dots, g_k)$ is

$$\mathcal{C}(g_1, \dots, g_n) := \left\{ g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n} \,\middle|\, \varepsilon_i \in \{0,1\} \right\}.$$

Observe that $\#\,\mathcal{C}(g_1, \dots, g_n) \leq 2^n$. We say $(g_1, \dots, g_k)$ is a *cubic* k-tuple if

$$\#\,\mathcal{C}(g_1, \dots, g_k) = 2^k.$$

Lemma 2.13. *Let G be a finitely generated group, and fix $S \in \Gamma_n(G)$. Let $\alpha > 1$ be a constant, and (k_i) be a log-dense sequence. Suppose for each $i \geq 1$, there is a path γ of length at most αk_i in $\Gamma_n(G)$, such that γ starts at S and visits some $S_1, \dots, S_{k_i} \in \Gamma_n(G)$ in that order. Suppose further that there is a cubic k_i-tuple $(g_1, \dots, g_{k_i})$, where $g_j \in S_j$ for each $1 \leq j \leq k_i$. Then $\Gamma_m(G,S)$ has exponential growth for every $m \geq n+1$.*

Proof. It is enough to show that $\Gamma_{n+1}(G,S)$ has exponential growth. Let $\Gamma = \Gamma_{n+1}(G)$, and $k = k_i$. By Lemma 2.1, it suffices to show that

$$\left|B_\Gamma(S^{(1)}, (\alpha+1)k)\right| \geq 2^k.$$

Given $(\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k$, we traverse the path γ in the first n coordinates of $\Gamma_{n+1}(G)$, but when we reach S_j, if $\varepsilon_j = 1$ we also apply a Nielsen transformation to multiply the last entry by g_j. This gives us a path γ' in $\Gamma_{n+1}(G)$ of length at most $\alpha k + k$. The path γ' ends at an element of $\Gamma_{n+1}(G)$ whose last entry is $g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n}$. Since $(g_1, \ldots, g_k)$ is cubic, there are 2^k distinct such elements. Thus, we have constructed 2^k distinct elements of $B_\Gamma(S^{(1)}, \alpha k + k)$, as desired. □

2.4. **Connectivity of product replacement graphs.** The connectivity of the product replacement graphs of a group G can be studied using the *Frattini subgroup* $\Phi(G)$, which is defined as the intersection of the maximal proper subgroups of G. We have

$$\Phi(G) = \left\{ g \in G \,\middle|\, \text{if } H \subsetneq G, \text{ then } \langle H, g \rangle \subsetneq G \right\}.$$

Informally, $\Phi(G)$ is the subgroup of *non-generators* of G, i.e. removing an element of $\Phi(G)$ from a generating set of G still leaves a generating set of G (see e.g. [Hall, §10.4]). It is easy to see that $\Phi(G)$ is a normal subgroup of G. We need the following connectivity result by Evans (see [Eva, Theorem 4.3]).

Theorem 2.14 (Evans)**.** *Suppose G is generated by some n-tuple. Let $m \geq n+1$, and suppose $\Gamma_m\big(G/\Phi(G)\big)$ is connected. Then $\Gamma_m(G)$ is connected.*

It is known that for any finite abelian group G with n generators, the product replacement graph $\Gamma_m(G)$ is connected for every $m > n$ [DG]. We use only the following special case, which is easy to verify by hand.

Lemma 2.15. *The product replacement graph $\Gamma_m(\mathbb{Z}_2^n)$ is connected for every $m \geq n$.*

In particular, suppose $G/\Phi(G) \cong \mathbb{Z}_2^n$. Then $\Gamma_m(G)$ is connected for every $m > n$.

3. General classes of groups

In this section, we prove that some general classes of groups have product replacement graphs of exponential growth

3.1. **Groups of exponential growth.** We prove that groups of exponential growth have exponentially growing product replacement graphs.

Proposition 3.1. *Let G be a finitely generated group, and let $S \in \Gamma_n(G)$. Suppose G has exponential growth. Then $\Gamma_m(G, S)$ has exponential growth for every $m \geq n+1$.*

Proof. It is enough to check that $\Gamma_{n+1}(G,S)$ has exponential growth. Let $S = (g_1, \ldots, g_n)$. Since G has exponential growth, there is an $\alpha > 1$, such that $B_{G,S}(r) = \{g \in G \mid \ell_S(g) \leq r\}$ has size at least α^r. For every $g \in B_{G,S}(S, r)$, we have that $(g_1, \ldots, g_n, g)$ is within r Nielsen moves of $S^{(1)} = (g_1, \ldots, g_n, 1)$. Hence, $\left|B_{\Gamma_{n+1}}(S^{(1)}, r)\right| \geq |B_{G,S}(r)| \geq \alpha^r$ for all $r \geq 0$. □

If G is a group of exponential growth generated by n elements it is not known whether $\Gamma_n(G)$ must have a connected component of exponential growth.

3.2. **Groups of polynomial growth.** Recall Gromov's Theorem:

Theorem 3.2 ([Gro])**.** *Every finitely generated group of polynomial growth is virtually nilpotent.*

This allows us to construct an element of infinite order in every group G of polynomial growth, which implies G has a product replacement graph of exponential growth.

Proposition 3.3. *Let G be a finitely generated group, and let $S \in \Gamma_n(G)$. Suppose G is infinite and has polynomial growth. Then $\Gamma_m(G, S)$ has exponential growth for every $m \geq n+2$.*

Proof. Note that it is enough to check that $\Gamma_{n+2}(G,S)$ has exponential growth. It is easy to see that G has an element of infinite order. Indeed, by Gromov's theorem, the group G has a nilpotent subgroup H of finite index. Then H is infinite and has a lower central series $H = H_0 > H_1 > \cdots > H_n = \{1\}$. Define i to be the smallest such that H_i/H_{i+1} is infinite. Then $[G : H_i]$ is finite, and therefore H_i is finitely generated. Thus, H_i/H_{i+1} is an infinite finitely generated abelian group, which implies that it has an element of infinite order. It follows that H_i has an element of infinite order, and thus so does G. By Lemma 2.8, we conclude that $\Gamma_{n+2}(G,S)$ has exponential growth. □

Note that by Milnor's theorem, every virtually solvable group G has either polynomial growth or exponential growth [Mil]. This implies:

Corollary 3.4. *Let G be a finitely generated infinite virtually solvable group. Then $\Gamma_m(G,S)$ has exponential growth, for every $m \geq n+2$.*

4. Automorphisms of the rooted binary tree

In this section, we introduce and discuss properties of the group $\mathrm{Aut}(\mathbf{T})$ of automorphisms of a binary tree.

4.1. **Definitions.** Let $\mathbf{T} = \{0,1\}^*$ denote the rooted binary tree consisting of finite strings over the alphabet $\{0,1\}$, whose root is the empty string, where the children of the string s are $s0$ and $s1$. Define $\mathrm{Aut}(\mathbf{T})$ to the group of automorphisms of this tree. Formally, $\mathrm{Aut}(\mathbf{T})$ consists of length preserving bijections g of $\mathbf{T}$ such that for any $s,t \in \mathbf{T}$, $g(st)$ begins with $g(s)$. To avoid confusion with the bit 1, we let $\mathbf{i} \in \mathrm{Aut}(\mathbf{T})$ denote the identity element. Let $g{\downarrow_s}$ denote the action of g on tails of strings beginning with s. In other words, we define it to satisfy $g(st) = g(s)g{\downarrow_s}(t)$.

Define $a \in \mathrm{Aut}(\mathbf{T})$ to be the automorphism which flips the first bit of s. Formally, $a(0s) = 1s$ and $a(1s) = 0s$ for all $s \in \mathbf{T}$. Clearly, every element of $\mathrm{Aut}(\mathbf{T})$ either fixes 0 and 1 or swaps them. Let g be an element that fixes them. Then $g(0s) = 0g{\downarrow_0}(s)$ and $g(1s) = 1g{\downarrow_1}(s)$. In this case, we write g in one of the following two forms, depending on which is more convenient:

$$g = (g{\downarrow_0}, g{\downarrow_1}) \quad \text{or} \quad g = \begin{pmatrix} g{\downarrow_0} \\ g{\downarrow_1} \end{pmatrix}.$$

On the other hand, suppose $g \in \mathrm{Aut}(\mathbf{T})$ swaps 0 and 1. Then $g = a(g{\downarrow_0}, g{\downarrow_1}) = (g{\downarrow_1}, g{\downarrow_0})a$. Thus, we can write every element $g \in \mathrm{Aut}(\mathbf{T})$ in the form $(h,k)a^\varepsilon$, for some $h,k \in \mathrm{Aut}(\mathbf{T})$ and some $\varepsilon \in \{0,1\}$. Moreover $(h,k)a = a(k,h)$ for all $h,k \in \mathrm{Aut}(\mathbf{T})$. In other words, we have $\mathrm{Aut}(\mathbf{T}) = (\mathrm{Aut}(\mathbf{T}) \times \mathrm{Aut}(\mathbf{T})) \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathrm{Aut}(\mathbf{T}) \times \mathrm{Aut}(\mathbf{T})$ by swapping the two coordinates.

Let $s \in \mathbf{T}$ be a given binary string. We say the *stabilizer* of s is the subgroup of $\mathrm{Aut}(\mathbf{T})$ consisting of those elements $g \in \mathrm{Aut}(\mathbf{T})$ which fix s:

$$\mathrm{Stab}(s) := \left\{ g \in \mathrm{Aut}(\mathbf{T}) \text{ s.t. } g(s) = s \right\}.$$

The n-th *level stabilizer* is the subgroup of $\mathrm{Aut}(\mathbf{T})$ consisting of those elements which fix the n-th level of $\mathbf{T}$:

$$\mathrm{Stab}_n := \bigcap_{s \in \{0,1\}^n} \mathrm{Stab}(s).$$

Let $g \in \mathrm{Stab}_n$. The *n-support* of g is

$$\mathrm{supp}_n(g) = \left\{ s \in \{0,1\}^n \text{ s.t. } g{\downarrow_s} \neq \mathbf{i} \right\}.$$

Finally, given $s \in \{0,1\}^n$, we define the *rigid stabilizer* of s to be the subgroup

$$\mathrm{Rist}(s) := \left\{ g \in \mathrm{Stab}_n \text{ s.t. } \mathrm{supp}_n(g) \subseteq \{s\} \right\}.$$

In other words, $\mathrm{Rist}(s)$ consists of those elements of $\mathrm{Aut}(\mathbf{T})$ which fix every string that does not begin with s.

For a subgroup G of $\mathrm{Aut}(\mathbf{T})$, define

$$\mathrm{Stab}_G(s) = G \cap \mathrm{Stab}(s) \quad \text{and} \quad \mathrm{Rist}_G(s) = G \cap \mathrm{Rist}(s).$$

Note that

$$\begin{aligned} &\mathrm{Rist}(0s) = \{(g, \mathbf{i}) \mid g \in \mathrm{Rist}(s)\} = \mathrm{Rist}(s) \times \{\mathbf{i}\}, \\ \text{and} \quad &\mathrm{Rist}(1s) = \{(\mathbf{i}, g) \mid g \in \mathrm{Rist}(s)\} = \{\mathbf{i}\} \times \mathrm{Rist}(s). \end{aligned}$$

4.2. **Growth in subgroups of** $\mathrm{Aut}(\mathbf{T})$**.** For distinct $s, s' \in \{0,1\}^m$, elements of $\mathrm{Rist}(s)$ and $\mathrm{Rist}(s')$ have disjoint n-support. We use Nielsen transformations to reach many of these elements. This implies we can find a large cubic set, which lets us construct many different generating n-tuples.

Lemma 4.1. *Let $G < \mathrm{Aut}(\mathbf{T})$ be finitely generated, and fix a generating n-tuple $S \in \Gamma_n(G)$. Suppose G acts transitively on every level of $\mathbf{T}$. Suppose there is a constant α such that for every $m \geq 1$, there is a string $s \in \{0,1\}^m$ and a nontrivial element $g \in \mathrm{Rist}_G(s)$ with $\ell(g) \leq \alpha 2^m$. Then $\Gamma_k(G,S)$ has exponential growth for every $k \geq n+2$*

Proof. Given m, define $L = \{0,1\}^m$ and $N = 2^m$. Fix $s \in L$ such that there is a nontrivial $g \in \mathrm{Rist}_G(s)$, satisfying $\ell(g) \leq \alpha N$. Since G acts transitively on L, we have that the Schreier graph $\mathrm{Schr}_S(G,L)$ is connected. Therefore, $\mathrm{Schr}_S(G,L)$ has a spanning tree $\mathcal{T}$. Consider a depth-first traversal of $\mathcal{T}$ with respect to the lexicographic order on L, starting at s. This is a path of length $2\,|L|-2 < 2N$ which visits every element of L. Suppose it visits them in the order $s_1, \ldots, s_N$. For each $1 \leq i \leq N$, define h_i to be the group element corresponding to the walk along this path from s to s_i, so that $(s_1, \ldots, s_N) = (h_1(s), \ldots, h_N(s))$. Then we have $\ell(h_2 h_1^{-1}) + \cdots + \ell(h_N h_{N-1}^{-1}) \leq 2N = 2^{m+1}$, and $\big(h_1(s), \ldots, h_N(s)\big)$ is a permutation of the elements of L.

Since $g \in \mathrm{Rist}_G(s)$, we have $h_i g h_i^{-1} \in \mathrm{Rist}_G\big(h_i(s)\big)$, for all $1 \leq i \leq N$. We claim that

$$(h_1 g h_1^{-1}, \ldots, h_N g h_N^{-1})$$

is a cubic N-tuple, i.e.

$$\#\left\{ \overrightarrow{\prod_{i=1\ldots N}} (h_i g h_i)^{\varepsilon_i}, \ \text{ where } \ \varepsilon_i \in \{0,1\} \right\} = 2^N.$$

Indeed, the surjection $\phi : \{0,1\}^N \to \mathcal{C}(h_1 g h_1^{-1}, \ldots h_N g h_N^{-1})$ given by

$$\phi(\varepsilon) := \overrightarrow{\prod_{i=1\ldots N}} (h_i g h_i^{-1})^{\varepsilon_i}$$

is also injective, since $\varepsilon_i = 1$ if and only if $s_i \in \mathrm{supp}_n \phi(\varepsilon)$. Hence $\#\,\mathcal{C}(h_1 g h_1^{-1}, \ldots h_N g h_N^{-1}) = 2^N$, as desired.

Since $\ell(g) \leq \alpha 2^m$, there is a path γ_1 in $\Gamma_{n+1}(G,S)$ of length at most $\alpha 2^m$

$$S^{(1)} = (g_1, \ldots, g_n, 1) \rightsquigarrow (g_1, \ldots, g_n, g) = (g_1, \ldots, g_n, h_1 g h_1^{-1}).$$

Observe that the distance in $\Gamma_{n+1}(G,S)$ between $(g_1, \ldots, g_n, h_i g h_i^{-1})$ and $(g_1, \ldots, g_n, h_{i+1} g h_{i+1}^{-1})$ is at most $2\ell(h_{i+1} h_i^{-1})$. Since $\ell(h_2 h_1^{-1}) + \cdots + \ell(h_N h_{N-1}^{-1}) \leq 2^{m+1}$, there is a path γ_2 in Γ_{n+1} of length at most 2^{m+2} which starts at $(g_1, \ldots, g_n, g)$ and visits each $(g_1, \ldots, g_n, h_i g h_i^{-1})$, in that order.

Composing γ_1 and γ_2, we see that there is a path in $\Gamma_{n+1}(G,S)$ of length at most $(\alpha+4)2^m$ which starts at $S^{(1)}$ and visits generating $(n+1)$-tuples containing $h_1 g h_1^{-1}, \ldots, h_N g h_N^{-1}$, in that order. These elements of G form a cubic 2^m-tuple. Applying Lemma 2.13 with $k_m = 2^m$, then, tells us that $\Gamma_k(G,S)$ has exponential growth for all $k \geq n+2$. □

Remark 4.2. We cannot replace $\ell(g) \leq \alpha 2^n$ in the hypotheses of this lemma with $\ell(g) \leq \alpha^n$ with some $\alpha > 2$. Roughly speaking, that would only let us reach a cubic 2^n-tuple in α^n steps. Thus, we can only generate an $r^{1/d}$-cube in $B_\Gamma(S,r)$, where $d = \log_2 \alpha$, which is not sufficient to guarantee exponential growth. We can, however, replace the assumption that $\ell(g) \leq \alpha 2^n$ with the assumption that we can reach a generating $(n+1)$-tuple containing g in $\alpha 2^n$ Nielsen moves.

5. The Grigorchuk group

5.1. **Definition.** The Grigorchuk group $\mathbb{G} < \mathrm{Aut}(\mathbf{T})$ is defined as $\mathbb{G} = \langle a, b, c, d \rangle$, where a flips the first bit of a string, and b, c, and d are defined recursively by the relations

$$\begin{aligned} b &:= (a, c) \\ c &:= (a, d) \\ d &:= (\mathbf{i}, b). \end{aligned}$$

It is easy to check that $a^2 = b^2 = c^2 = d^2 = bcd = \mathbf{i}$. Thus, $\mathbb{G}$ is actually generated by just three elements: $\mathbb{G} = \langle a, b, c \rangle$.

Here is an explicit description of the action of these involutions on $\mathbf{T}$.

$$\begin{aligned} d(1^n) &= 1^n \\ d(1^n 0 s) &= \begin{cases} 1^n 0 s, & n \equiv 0 \pmod 3 \\ 1^n 0 a(s), & n \equiv 1, 2 \pmod 3 \end{cases} \end{aligned}$$

In other words, d changes at most one bit in a string – the bit after the first 0. Specifically, d flips that bit if and only if the number n of 1's in the string up to that point is 1 or 2 $\pmod 3$. Similarly, c flips it when $n \equiv 0, 2 \pmod 3$, and b flips it when $n \equiv 0, 1 \pmod 3$.

Theorem 5.1 (Gigorchuk)**.** *The group $\mathbb{G}$ has intermediate growth.*

The theorem was first proved by Grigorchuk in [G1] (see also [GP, dlH1]).

5.2. **Connectivity of $\Gamma_n(\mathbb{G})$.** We prove the following result:

Proposition 5.2. *For each $n \geq 4$, the product replacement graph $\Gamma_n(\mathbb{G})$ is connected (see also §7.3).*[1]

Proof. Fix $n \geq 4$. It is known that $\mathbb{G}/\Phi(\mathbb{G}) \cong \mathbb{Z}_2^3$ (see [Per] and [G2, §6]). The graph $\Gamma_n(\mathbb{Z}_2^3)$ is connected by Lemma 2.15. Thus, by Lemma 2.14, $\Gamma_n(\mathbb{G})$ is connected. □

5.3. **Exponential growth in $\Gamma_n(\mathbb{G})$.** The goal of this section is to prove the following result:

Theorem 5.3. *For each $n \geq 5$, the product replacement graph $\Gamma_n(\mathbb{G})$ of the Grigorchuk group has exponential growth.*

The proof is based on Lemma 4.1. Roughly, our strategy is to find an element g of $\mathrm{Rist}_{\mathbb{G}}(1^n)$ with length $O(2^n)$. In $O(2^n)$ more steps, we conjugate g to reach an element of $\mathrm{Rist}_{\mathbb{G}}(s)$ for each s on the same level of $\mathbf{T}$. Then we can construct every product of these conjugates in $O(2^n)$ steps. There are 2^{2^n} such products, which gives us exponential growth.

Proof of Theorem 5.3. Fix $n \geq 5$. It is easy to check that $\mathbb{G}$ acts transitively on the levels of $\mathbf{T}$ (see e.g. [dlH1, §VIII] or Lemma 6.1, below). By Lemma 4.1, it suffices to show that for every $m \geq 0$, there is a nontrivial element of $\mathrm{Rist}(1^m)$ of length at most $16 \cdot 2^m$ with respect to the generating 3-tuple (a, b, c).

Define $t_0 = abab$. Observe that $t_0^2(111) = 110$, and therefore $t_0^2 \neq \mathbf{i}$. We prove by induction on m that there is a $t_m \in \mathbb{G}$ of the form

$$t_m = \overrightarrow{\prod_{i=1\ldots N}} abax_i, \qquad (*)$$

where $N = 2^m$, $x_i \in \{b, c, d\}$ for each $1 \leq i \leq 2^m$, such that $t_m^2 \in \mathrm{Rist}_{\mathbb{G}}(1^m)$ and $t_m{\downarrow_{1^m}} = t_0$. The base case $m = 0$ is trivial.

[1] After this paper was written, we learned that the proposition was independently derived in [Myr].

Given t_m and (x_i) related by $(*)$, for each $0 \le i \le N$ we define $x_i' \in \{b, c, d\}$ by $x_i' = (a^{\varepsilon_i}, x_i)$ where $\varepsilon_i \in \{0, 1\}$. We define t_{m+1} by applying the rewriting rules $a \mapsto aba$, $b \mapsto d$, $c \mapsto b$, $d \mapsto c$ to t_m. Then we have

$$t_{m+1} = \left[\overrightarrow{\prod_{i=1\ldots N}} (aba)d(aba)x_i'\right] = \left[\overrightarrow{\prod_{i=1\ldots N}} \binom{c}{a}\binom{\mathbf{i}}{b}\binom{c}{a}\binom{a^{\varepsilon_i}}{x_i}\right] = \binom{a^{\varepsilon}}{t_m},$$

Thus,

$$t_{m+1}^2 = (\mathbf{i}, t_m^2) \in \{\mathbf{i}\} \times \mathrm{Rist}(1^m) = \mathrm{Rist}(1^{m+1}),$$

and $t_{m+1}^2{\downarrow_{1^{m+1}}} = t_m^2{\downarrow_{1^m}} = t_0^2$.

Since $t_m^2{\downarrow_{1^m}} = t_0^2 \ne \mathbf{i}$, we can conclude that $t_m^2 \ne \mathbf{i}$. Hence, for every $m \ge 0$, we have that t_m^2 is a nontrivial element of $\mathrm{Rist}_{\mathbb{G}}(1^m)$, with $\ell_{\langle a,b,c\rangle}(t_m^2) \le 2\ell_{\langle a,b,c,d\rangle}(t_m^2) \le 16 \cdot 2^m$, which concludes the proof. $\square$

5.4. **No elements of large order.** We note that Proposition 2.10 cannot be used to prove Theorem 5.3. To use it, we would need to find elements whose order is exponential in their length. However, such elements do not exist by the following result.

Proposition 5.4. *There are constants β and d, such that for any $g \in \mathbb{G}$ with $g \ne 1$, we have*

$$\mathrm{ord}(g) \le \beta \ell(g)^d.$$

We derive the proposition from the following Cancellation Lemma, which is a key step in the proof of Theorem 5.1.[2]

Lemma 5.5 (Cancellation Lemma)**.** *There are constants $\lambda < 1$ and $\alpha > 0$ such that for any $g \in \mathbb{G}$,*

$$\ell(g{\downarrow_{000}}) + \ell(g{\downarrow_{001}}) + \cdots + \ell(g{\downarrow_{111}}) \le \lambda \ell(g) + \alpha.$$

Proof of Proposition 5.4. Define $f(n) = \max\{\mathrm{ord}(g) \mid g \in \mathbb{G}, \ell(g) \le n\}$.

By Lemma 5.5, for every $g \in \mathbb{G}$ with $\ell(g) \le n$, we have

$$\ell(g{\downarrow_{000}}) + \ell(g{\downarrow_{001}}) + \cdots + \ell(g{\downarrow_{111}}) \le \lambda n + \alpha.$$

Let s be a length 3 bitstring, and consider the orbit $s, g(s), g^2(s), \ldots, g^k(s) = s$ under the action of g. We have

$$\begin{aligned} g^k(st) = g^{k-1}\big(g(s) g{\downarrow_s}(t)\big) &= g^{k-2}\big(g^2(s)(g{\downarrow_{g(s)}}\, g{\downarrow_s})(t)\big) \\ = \quad \ldots \quad &= s(g{\downarrow_{g^{k-1}(s)}} \cdots g{\downarrow_{g(s)}}\, g{\downarrow_s})(t). \end{aligned}$$

Hence, $g^k{\downarrow_s} = g{\downarrow_{g^{k-1}(s)}} \cdots g{\downarrow_{g(s)}}\, g{\downarrow_s}$, and therefore $\ell(g^k{\downarrow_s}) \le \lambda n + \alpha$. Thus $\mathrm{ord}(g^k{\downarrow_s}) \le f(\lambda n + \alpha)$. Since k is a factor of 8, we can conclude that $\mathrm{ord}(g^8{\downarrow_s}) \le f(\lambda n + \alpha)$. We know that $g^8 \in \mathrm{Stab}_3$, and each $\mathrm{ord}(g^8{\downarrow_s})$ is a power of 2. This implies

$$\begin{aligned} \mathrm{ord}(g^8) &= \mathrm{lcm}\big(\mathrm{ord}(g^8{\downarrow_{000}}), \ldots, \mathrm{ord}(g^8{\downarrow_{111}})\big) \\ &= \max\big(\mathrm{ord}(g^8{\downarrow_{000}}), \ldots, \mathrm{ord}(g^8{\downarrow_{111}})\big) \le f(\lambda n + \alpha). \end{aligned}$$

Thus, $\mathrm{ord}(g) \le 8f(\lambda n + \alpha)$ for all g with $\ell(g) \le n$. Hence $f(n) \le 8f(\lambda n + \alpha)$. For any λ' with $\lambda < \lambda' < 1$, we can find n_0 such that $\lambda' n \ge \lambda n + \alpha$ for all $n > n_0$. It follows that $f(n) \le 8f(\lambda' n)$ for all $n > n_0$.

Now define $\beta = f(n_0)$ and $d = -\log(8)/\log \lambda'$. We prove by induction on n that $f(n) \le \beta n^d$. The base cases are $1 \le n \le n_0$. In that case, $f(n) \le f(n_0) \le \beta \le \beta n^d$. For the inductive step, we have $n > n_0$. Then

$$f(n) \le 8f(\lambda' n) = 8f(\lfloor \lambda' n \rfloor) \le 8\beta \lfloor \lambda' n \rfloor^d \le 8\beta(\lambda' n)^d = \beta n^d.$$

$\square$

[2]The Cancellation Lemma in [GP] assumes that g stabilizes the first three levels of $\mathbf{T}$, but this form follows easily.

6. The generalized Grigorchuk groups

In this section, we use the same approach to analyze growth in the product replacement graph of $\mathbb{G}_\omega$. The same techniques apply, but the technical details are more involved.

6.1. **Definition.** Let ω be an infinite string in the alphabet[3] $\{b, c, d\}$. The *generalized Grigorchuk group* $\mathbb{G}_\omega$ is the group of automorphisms of $\{0,1\}^n$ given by $\mathbb{G}_\omega = \langle a, b_0, c_0, d_0\rangle$. Here, the element a flips the first digit of a string, and for each $x \in \{b, c, d\}$, the elements x_n are defined recursively by

$$x_n := (a^\varepsilon, x_{n+1}), \qquad \text{where } \varepsilon = \begin{cases} 0, & x = \omega_n \\ 1, & \text{otherwise.} \end{cases}$$

For convenience, we write $b = b_0$, $c = c_0$, and $d = d_0$. As with $\mathbb{G}$, we have $a^2 = b^2 = c^2 = d^2 = bcd = 1$.

As before, we give a more explicit description of the action of $\mathbb{G}_\omega$ on $\mathbf{T}$. Given $x \in \{b, c, d\}$ and $s \in \mathbf{T}$,

$$x(1^n) = 1^n, \text{ and}$$
$$x(1^n 0s) = \begin{cases} 1^n 0s, & \omega_n = x \\ 1^n 0a(s), & \text{otherwise.} \end{cases}$$

Taking $\omega = dcbdcbdcbdcb\ldots$ gives the usual Grigorchuk group. The following fact is well-known, but we include a proof here for completeness.

Lemma 6.1. *The generalized Grigorchuk group $\mathbb{G}_\omega$ acts transitively on every level of $\mathbf{T}$.*

Proof. We prove that $\mathbb{G}_\omega$ acts transitively on the n-th level by induction on n. This is trivial for $n = 0$, and true for $n = 1$ because $a \in \mathbb{G}_\omega$. For $n > 1$, note that it suffices to show that for each $s \in \{0,1\}^n$, there is a $g \in \mathbb{G}_\omega$ such that $g(s) = 1^{n-2}00$. Consider $s \in \{0,1\}^n$. We know that $s = s'd$, for some $s' \in \{0,1\}^{n-1}$ and $d \in \{0,1\}$. By the induction hypothesis, $\mathbb{G}_\omega$ acts transitively on $\{0,1\}^{n-1}$. Thus there is a $g \in \mathbb{G}_\omega$ with $g(s') = 1^{n-2}0$. Then either $g(s) = 1^{n-2}00$ or $g(s) = 1^{n-2}01$. In the latter case, there is an $x \in \{b, c, d\}$ such that $\omega_{n-2} \neq x$, and then $x(g(s)) = 1^{n-2}00$. In both cases, there is an $h \in \mathbb{G}_\omega$ with $h(s) = 1^{n-2}00$. □

6.2. **Exponential growth in $\Gamma_n(\mathbb{G}_\omega)$.** We prove the following result.

Theorem 6.2. *Let $\mathbb{G}_\omega$ be a generalized Grigorchuk group. Then $\Gamma_n(\mathbb{G}_\omega)$ is connected for each $n \geq 4$, and has exponential growth for each $n \geq 5$.*

First, we need some lemmas about $\mathbb{G}_\omega$. A standard computation shows that, under some weak assumptions on ω, every element of $\mathbb{G}_\omega$ has finite order. We will use the following more specialized result.

Lemma 6.3. *Suppose $\omega_{n-1} = d$. Then in $\mathbb{G}_\omega$, we have $(ad_k)^{2^{n-k+1}} = \mathbf{i}$ for every $0 \leq k < n$.*

Proof. Since $\omega_{n-1} = d$, we have $d_{n-1} = (\mathbf{i}, d_n)$ and $ad_{n-1}a = (d_n, \mathbf{i})$. We prove the lemma by induction on $j = n - k$. When $j = 1$, i.e. $k = n - 1$, we have

$$(ad_k)^4 = [(ad_{n-1}a)d_{n-1}]^2 = \left[\begin{pmatrix} d_n \\ \mathbf{i} \end{pmatrix}\begin{pmatrix} \mathbf{i} \\ d_n \end{pmatrix}\right]^2 = \begin{pmatrix} d_n^2 \\ d_n^2 \end{pmatrix} = \mathbf{i}.$$

When $j > 1$, i.e. $k < n - 1$, the induction hypothesis tells us $(ad_{k+1})^{2^j} = \mathbf{i}$. Note that also $(d_{k+1})^{2^j} = \mathbf{i}$, since d_{k+1} has order 2. Then, for some $\varepsilon \in \{0, 1\}$, we have

$$(ad_k)^{2^{j+1}} = [(ad_k a)d_k]^{2^j} = \left[\begin{pmatrix} d_{k+1} \\ a^\varepsilon \end{pmatrix}\begin{pmatrix} a^\varepsilon \\ d_{k+1} \end{pmatrix}\right]^{2^j} = \begin{pmatrix} (a^\varepsilon d_{k+1})^{-2^j} \\ (a^\varepsilon d_{k+1})^{2^j} \end{pmatrix} = \mathbf{i}.$$

□

[3]The usual definition uses the alphabet $\{0, 1, 2\}$ but for our purposes it is more convenient to use $\{b, c, d\}$.

Lemma 6.4. *Suppose $\omega \in \{b, c, d\}^*$ is not eventually constant. Then for each $n \geq 0$, there is a nontrivial $t \in \mathrm{Rist}_{\mathbb{G}_\omega}(1^n)$ with $\ell(t) \leq 2^{n+2}$.*

Proof. This is trivial if $n = 0$. If $n > 1$, then by relabeling b, c, and d if necessary, we may assume $\omega_{n-1} = d$.

By induction on $j = n - k$ we show that for every $0 \leq k \leq n$, there is a t_k of the form

$$t_k = \overrightarrow{\prod_{i=1\ldots 2^{n-k}}} a x_i,$$

where $x_i \in \{b_k, d_k\}$ for each i, and there is an odd number of i's with $x_i = d_k$, such that $t_k^2 \in \mathrm{Rist}(1^{n-k})$, and $t_k^2 \neq \mathbf{i}$.

For $j = 0$, i.e. $k = n$, we define $t_n = ad_n$. We know that $d_n = (a^\varepsilon, d_{n+1})$ for some $\varepsilon \in \{0, 1\}$, and therefore we have

$$t_n^2 = (ad_n a)d_n = \begin{pmatrix} d_{n+1} \\ a^\varepsilon \end{pmatrix} \begin{pmatrix} a^\varepsilon \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} d_{n+1} a^\varepsilon \\ (d_{n+1} a^\varepsilon)^{-1} \end{pmatrix}.$$

Since ω is not eventually constant, we know that there is an $m \geq n + 1$ with $\omega_m \neq d$. Therefore we have

$$d_{n+1}(1^{m-n-1}00) = 1^{m-n-1} d_m(00) = 1^{m-n-1} 0a(0) = 1^{m-n-1}01.$$

Hence, $d_{n+1} \neq \mathbf{i}$. It follows that $d_{n+1}a^\varepsilon$ is nontrivial whether $\varepsilon = 0$ or $\varepsilon = 1$. Therefore, $t_n^2 \neq \mathbf{i}$.

For $j = 1$, i.e. $k = n - 1$, we define $t_k = ab_{n-1}ad_{n-1}$. We have $\omega_k = d$, hence $d_k = (\mathbf{i}, d_{k+1})$ and $b_k = (a, b_{k+1})$. Therefore, we have:

$$t_{n-1}^2 = [(ab_{n-1}a)d_{n-1}]^2 = \left[\begin{pmatrix} b_n \\ a \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ d_n \end{pmatrix} \right]^2 = \begin{pmatrix} \mathbf{i} \\ (ad_n)^2 \end{pmatrix} = \begin{pmatrix} \mathbf{i} \\ t_n^2 \end{pmatrix}.$$

For $j > 1$, i.e. $k < n - 1$, let $N = 2^{n-k-1}$. We have

$$t_{k+1} = \overrightarrow{\prod_{i=1\ldots N}} a x_i$$

from the previous step. For each $1 \leq i \leq N$, we know that $x_i = b_{k+1}$ or d_{k+1}, and we define

$$x_i' = \begin{cases} b_k, & x_i = b_{k+1} \\ d_k, & x_i = d_{k+1}. \end{cases}$$

Then $x_i' = (a^{\varepsilon_i}, x_i)$ for some $\varepsilon_i \in \{0, 1\}$. We have three possibilities:

Case (i): $\omega_k = b$. Then $d_k = (a, d_{k+1})$ and $b_k = (\mathbf{i}, b_{k+1})$. Define

$$t_k = \overrightarrow{\prod_{i=1\ldots N}} a d_k a x_i'.$$

The product has an even number of terms, thus, we have not changed the parity of the number of d_k's in the product, which implies it is still odd.

$$t_k = \overrightarrow{\prod_{i=1\ldots N}} (ad_k a)x_i' = \overrightarrow{\prod_{i=1\ldots N}} \begin{pmatrix} d_{k+1} \\ a \end{pmatrix} \begin{pmatrix} a^{\varepsilon_i} \\ x_i \end{pmatrix} = \begin{pmatrix} \overrightarrow{\prod} d_{k+1} a^{\varepsilon_i} \\ t_{k+1} \end{pmatrix},$$

where the final product runs over $i = 1 \ldots N$. Observe that $\varepsilon_i = 1$ if and only if $x_i = d_i$. There are an odd number of such i, therefore $\overrightarrow{\prod} d_{k+1} a^{\varepsilon_i}$ is a product containing an odd number of a's and an even number of d_{k+1}'s. The elements d_{k+1} and a have order 2, so the group $\langle d_{k+1}, a \rangle$ is a dihedral group in which they are both reflections. Hence, the product $\overrightarrow{\prod} d_{k+1} a^{\varepsilon_i}$ is also a reflection in that dihedral group, and thus it has order 2. Therefore, $t_k^2 = (\mathbf{i}, t_{k+1}^2)$.

Case (ii): $\omega_k = d$. Then $d_k = (\mathbf{i}, d_{k+1})$ and $b_k = (a, b_{k+1})$. Define

$$t_k = \overrightarrow{\prod_{i=1\ldots N}} ab_k a x'_i,$$

and argue as in case (i).

Case (iii): $\omega_k = c$. Then $d_k = (a, d_{k+1})$ and $b_k = (a, b_{k+1})$. Hence $x'_i = (a, x_i)$ for each $0 \le i \le N$. We can again define

$$t_k = \overrightarrow{\prod_{i=1\ldots N}} ad_k a x'_i.$$

Then

$$t_k = \overrightarrow{\prod_{i=1\ldots N}} (ad_k a)x'_i = \overrightarrow{\prod_{i=1\ldots N}} \binom{d_{k+1}}{a}\binom{a}{x_i} = \binom{(d_{k+1}a)^N}{t_{k+1}}.$$

By Lemma 6.3, we have $(d_{k+1}a)^{2N} = (d_{k+1}a)^{2^{n-k}} = \mathbf{i}$. Hence, $t_k^2 = (\mathbf{i}, t_{k+1}^2)$.

In all three cases, $t_k^2 = (\mathbf{i}, t_{k+1}^2)$. It follows that t_k^2 is nontrivial and

$$t_k^2 \in \{\mathbf{i}\} \times \mathrm{Rist}(1^{n-k-1}) = \mathrm{Rist}(1^{n-k}).$$

Thus, we have a nontrivial $t_0^2 \in \mathrm{Rist}(1^n) \cap \mathbb{G}_\omega$, with $\ell(t_0^2) \le 2^{n+2}$, as desired. $\square$

Proof of Theorem 6.2. It is known that $\mathbb{G}_\omega/\Phi(\mathbb{G}_\omega) \cong \mathbb{Z}_2^k$ for some $k \le 3$ [Per], [G2, §6]. We know $\Gamma_n(\mathbb{Z}_2^k)$ is connected by Lemma 2.15. Lemma 2.14 tells us that $\Gamma_n(\mathbb{G}_\omega)$ is connected for each $n \ge 4$.

If ω is eventually constant, then it is not hard to check that $\mathbb{G}_\omega$ has polynomial growth. In fact, it is virtually abelian [G2, §2]. The group $\mathbb{G}_\omega$ is generated by three elements, $\mathbb{G}_\omega = \langle a, b, c\rangle$. Hence, by Proposition 3.3, the product replacement graph $\Gamma_n(\mathbb{G}_\omega)$ has exponential growth for each $n \ge 5$.

Otherwise, for every $m \ge 0$, Lemma 6.4 gives a nontrival $t \in \mathrm{Rist}_{\mathbb{G}_\omega}(1^m)$ of length at most $4 \cdot 2^m$. Since $\mathbb{G}_\omega$ acts transitively on the levels of $\mathbf{T}$, we can apply Lemma 4.1 to conclude that $\Gamma_6(\mathbb{G}_\omega)$ has exponential growth from $(a, b, c, d, 1, 1)$.

In fact, the group $\mathbb{G}_\omega$ is actually generated by (a, b, c), and rewriting t as a word in these generators at most doubles its length. Thus, we also have that $\Gamma_5(\mathbb{G}_\omega)$ has exponential growth from $(a, b, c, 1, 1)$. It follows that $\Gamma_n(\mathbb{G}_\omega)$ has exponential growth for each $n \ge 5$. $\square$

7. Final remarks

7.1. The motivation behind our Main Conjecture in the Introduction is rather interesting, which makes the conjecture both natural and speculative. First, recall that $\Gamma_k(G)$ are Schreier graphs of $\mathrm{Aut}(F_k)$, generated by Nielsen transformations [LP] (see also [LŻ, P1]). A well known conjecture states that $\mathrm{Aut}(F_k)$ has *Kazhdan's property* (T) for $k > 3$. If true, this would imply the following conjecture:

Conjecture 7.1. *For every infinite group G, product replacement graphs $\Gamma_k(G)$ are* non-amenable*, for k large enough.*

In particular, this conjecture implies that all connected components of $\Gamma_k(G)$ are infinite and have exponential growth, for all G and k large enough. We should mention that $\mathrm{Aut}(F_k)$ does not have (T) for $k = 2$ and 3 (see [GL, Lub]). On the other hand, the non-amenability of $\Gamma_n(G)$ follows from a weaker property (τ) for an appropriate family of subgroups (see [LŻ]).

Note that our Conjecture 1.6 is a weaker version of this claim. Here we accounted for the possibility that there can be many connected components, and are working with only one of them. Our Main Conjecture is even weaker; implicit in it is a reference to a conjecture that every generating k-tuple is connected to a redundant generating k-tuple in $\Gamma_k(G)$. For this and stronger conjectures on connectivity of $\Gamma_k(G)$, see [P1] (see also [BKM]).

7.2. In a followup paper [M1], the first author establishes Conjecture 7.1 for several classes of groups of exponential growth, which include virtually solvable groups (this an extension of Theorem 3.4), linear groups, random finitely presented groups (in Gromov sense), and hyperbolic groups. We use a technical extension of *uniform exponential growth* and *uniform non-amenability* (see [A+, BG, dlH2, Wil]).

Unfortunately, the explicit combinatorial approach in this paper, does not seem to be strong enough to establish Conjecture 7.1 for Grigorchuk group, which we state as a separate conjecture of independent interest.

Conjecture 7.2. *Product replacement graphs* $\Gamma_k(\mathbb{G})$ *are non-amenable, for all* $k \geq 5$.

7.3. There are several other directions in which our Theorem 5.3 can be extended. First, there is the problem of smaller k: we believe that that $\Gamma_3(\mathbb{G})$ is connected (cf. Lemma 2.15 and Proposition 5.2).[4] Moreover, it is conceivable that both $\Gamma_3(\mathbb{G})$ and $\Gamma_4(\mathbb{G})$ have exponential growth, the cases missing from Theorem 5.3.

Similarly, in case Conjecture 7.2 proves too difficult, there is a weaker and perhaps more accessible open problem.

Conjecture 7.3. *The nearest neighbor random walk on* $\Gamma_k(\mathbb{G})$ *has positive speed, for all* $k \geq 5$.

The speed of r.w. is defined as the limit of $\mathbb{E}[\mathrm{dist}(t)/t]$ as $t \to \infty$, where $\mathrm{dist}(t)$ is the distance of the r.w. after t steps, from the starting vertex. It is known that non-amenable graphs have positive speed, but so do some amenable graphs, such as the standard Cayley graph of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}^3$ (see e.g. [Pete, Woe]). We believe it might be possible to extend our approach to establish the positive speed of r.w. on $\Gamma_k(\mathbb{G})$, and we intend to return to this problem.

7.4. The techniques in this paper generalize fairly easily to several other groups of intermediate growth, such as the Gupta–Sidki p-groups [GS], as well as large families of Grigorchuk p-groups. Many groups of intermediate growth, such as the groups of oscillating growth defined in [KP], have, by construction, some $\mathbb{G}_\omega$ as a subgroup or a factor group. Such groups, then, also have exponential Nielsen growth, by Proposition 2.7.

In fact, the techniques in this paper apply to a general class of branch groups defined in [Bar] called *splitter-mixer groups*. Many known group of intermediate growth appears to be based on a splitter-mixer group (see, however, [Nek]). The proof will appear in [M2].

In summary, although we have yet to find proofs in all cases, we believe the Main Conjecture holds for all *known* constructions of groups of intermediate growth. In that sense the situation is similar to the "$p_c < 1$" conjecture by Benjamini and Schramm [BS] for groups of superlinear growth. The conjecture is known to hold for groups of exponential and polynomial growth, and by an ad hoc argument for Grigorchuk groups and general self-similar groups [MP]. It remains open for general groups of intermediate growth (see [Pete]).

7.5. Lemma 2.14 is an analogue for infinite groups of the following result in [LP] (see also [P1]). Let G and H be finite groups with k generators, and $f : G \to H$ is a surjective group homomorphism, then the extension $f : \Gamma_k(G) \to \Gamma_k(H)$ is surjective. That is, every generating k-tuple of H lifts to a generating k-tuple of G. As a corollary, if $\Gamma_k(G)$ is connected, then so is $\Gamma_k(H)$.

This claim is not true of infinite groups. For example, let $G = F_k$ and H be any group such that $\Gamma_k(H)$ has more than two connected components. Then there is no surjective map $\Gamma_k(G) \to \Gamma_k(H)$ which sends neighbors to neighbors, since such a map can only cover two connected components of $\Gamma_k(H)$.

7.6. The connectivity of product replacement graphs is delicate already for finite groups. For example, Dunwoody showed in [Dun], that if G is a finite solvable group with d generators, then $\Gamma_k(G)$ is connected, for every $k > d$ (see also [P1]). This property is conjectured to hold for all finite groups, but fails for infinite groups, even for metabelian groups (see [P1] and references therein).

[4] See also Corollary 1.2 and Question 1 in [Myr].

As of now, is unknown whether for any finitely generated group G, graphs $\Gamma_k(G)$ are connected for all sufficiently large k. It is not even known that if $\Gamma_k(G)$ is connected then $\Gamma_{k+1}(G)$ is connected. The difficulty arises from the possibility that $\Gamma_{k+1}(G)$ has a connected component which consists of non-redundant generating $(k+1)$-tuples. However, it is not hard to check that in $\Gamma_{2d}(G)$ every element of the form $(g_1, \ldots, g_d, 1, \ldots, 1)$ lies in the same connected component, which we may call $\Gamma^\star_{2d}(G)$. Then if we know that some connected component of $\Gamma_d(G)$ has exponential growth, we know that $\Gamma^\star_{2d}(G)$ has exponential growth.

7.7. Proposition 2.7 relates the Nielsen growth of a subgroup of G to the Nielsen growth of G. We conjecture that if G has exponential Nielsen growth, then so does every finite index subgroup of G. This would imply that the property of having exponential Nielsen growth respects virtual isomorphism. More generally, it would be interesting to see if this property is an invariant under quasi-isometry.

7.8. Although we proved Theorem 6.2 without the use of Proposition 2.10, there are in fact many sequences ω to which the proposition applies. For example, let $n_0 = 1$, and $n_{r+1} = n_r + 2^{n_r}$. Consider the group $\mathbb{G}_\omega$ where

$$\omega_n = \begin{cases} b, & n = n_{2r} \text{ for some } r \geq 0 \\ d, & n = n_{2r+1} \text{ for some } r \geq 0 \\ c, & \text{otherwise.} \end{cases}$$

For each $n \geq 0$, either b or d does not occur in the length 2^n substring of ω starting at ω_n. It is easy to check that then either ab_n or ad_n has order at least 2^{2^n}, and there is a $g \in \mathrm{Stab}_{\mathbb{G}_\omega}(1^n)$ of length $O(2^n)$ with $g\!\downarrow_{1^n} = ab_n$ or ad_n, so the proposition applies. We omit the details.

7.9. Finally, let us mention that the notion of exponential Nielsen growth may be applicable to sequences of finite groups, which stabilize in a certain sense. Proving such a result would be a step towards proving expansion of product replacement graphs of general finite groups (see [P1, P2]). We refer to [Bla] for the notion of growth of finite groups, and to [Ell] for a recent conceptual approach.

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References

[Adi] S. Adian, *The Burnside problem and identities in groups*, Springer, Berlin, 1979.

[A+] G. N. Arzhantseva, J. Burillo, M. Lustig, L. Reeves, H. Short and E. Ventura, Uniform non-amenability, *Adv. Math.* **197** (2005), 499–522

[Bar] L. Bartholdi, Groups of intermediate growth, `arXiv:0201293`.

[BL] H. Bäärnhielm and C. R. Leedham-Green, The product replacement prospector, *J. Symbolic Comput.* **47** (2012), 64–75.

[BS] I. Benjamini and O. Schramm, Percolation beyond $\mathbb{Z}^d$, many questions and a few answers, *Electron. Comm. Probab.* **1** (1996), 71–82.

[Bla] S. Black, Asymptotic growth of finite groups, *J. Algebra* **209** (1998), 402–426.

[BKM] A. V. Borovik, E. I. Khukhro and A. G. Myasnikov, The Andrews-Curtis Conjecture and Black Box Groups, *Int. J. Algebra Comput.* **13** (2003), 415–436.

[BLM] A. V. Borovik, A. Lubotzky and A. G. Myasnikov, The finitary Andrews-Curtis conjecture, in *Progr. Math.* **248**, Birkhäuser, Basel, 2005, 15–30.

[BG] E. Breuillard and T. Gelander, Uniform independence in linear groups, *Invent. Math.* **173** (2008), 225–263.

[dlH1] P. de la Harpe, *Topics in Geometric Group Theory*, University of Chicago Press, Chicago, 2000.

[dlH2] P. de la Harpe, Uniform growth in groups of exponential growth, *Geom. Dedicata* **95** (2002), 1–17.

[DG] P. Diaconis and R. Graham, The graph of generating sets of an abelian group, *Colloq. Math* **80** (1999), 31–38.

[Dun] M. J. Dunwoody, Nielsen transformations, in *Computational Problems in Abstract Algebra*, Pergamon Press, Oxford, 1970, 45–46.

[Ell] J. S. Ellenberg, Superstrong approximation for monodromy groups, `arXiv:1210.3757`.

[Eva] M. J. Evans, Presentations of groups involving more generators than are necessary, *Proc. LMS* **67** (1993), 106–126.

[G1] R. I. Grigorchuk, Degrees of growth of finitely generated groups, and the theory of invariant means, *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), 939–985.

[G2] R. I. Grigorchuk, Solved and unsolved problems around one group, in *Infinite Groups: Geometric, Combinatorial and Dynamical Aspects*, Birkhäuser, Basel, 2005, 117–218.

[G3] R. I. Grigorchuk, Some problems of the dynamics of group actions on rooted trees, *Proc. Steklov Inst. Math.* **273** (2011), 64-175.

[GP] R. I. Grigorchuk and I. Pak, Groups of intermediate growth, an introduction, *L'Ens. Math.* **54** (2008), 251–272.

[Gro] M. Gromov, Groups of polynomial growth and expanding maps, *IHES Publ. Math.* **53** (1981), 53–78.

[GL] F. Grunewald and A. Lubotzky, Linear representations of the automorphism group of a free group, *Geom. Funct. Anal.* **18** (2009), 1564–1608.

[GS] N. Gupta and S. Sidki, On the Burnside problem for periodic groups, *Math. Z.* **182** (1983), 385–388.

[Hall] M. Hall, *The Theory of Groups*, Chelsea, New York, 1976.

[KP] M. Kassabov and I. Pak, Groups of oscillating intermediate growth, *Ann. Math.* **177** (2013), 1113–1145.

[Lub] A. Lubotzky, *Discrete Groups, Expanding Graphs, and Invariant Measures*, Birkhäuser, Basel, 1994.

[LP] A. Lubotzky and I. Pak, The product replacement algorithm and Kazhdan's property (T), *J. AMS* **14** (2001), 347–363.

[LŻ] A. Lubotzky and A. Żuk, *On property* (τ), monograph in preparation.

[M1] A. Malyshev, Non-amenability of product replacement graphs, in preparation.

[M2] A. Malyshev, *Combinatorics of finitely generated groups*, Ph.D. thesis, UCLA, in preparation.

[Met] W. Metzler, On the Andrews-Curtis conjecture and related problems, in *Contemp. Math.* **44**, AMS, Providence, RI, 1985, 35–50.

[Mil] J. Milnor, Growth of finitely generated solvable groups, *J. Diff. Geom.* **2** (1968), 447–449.

[MP] R. Muchnik and I. Pak, Percolation on Grigorchuk groups, *Comm. Algebra* **29** (2001), 661–671.

[Myr] A. Myropolska, Andrews–Curtis and Nielsen equivalence relations on some infinite groups, `arXiv:`1304.2668.

[Nek] V. Nekrashevych, *Self-similar Groups*, AMS, Providence, RI, 2005.

[NP] A. C. Niemeyer and C. E. Praeger, Complexity and computation in matrix groups, in *Aspects of Complexity*, de Gruyter, Berlin, 2001, 87–113.

[P1] I. Pak, What do we know about the product replacement algorithm?, in *Groups and Computation III*, de Gruyter, Berlin, 2001, 301–347.

[P2] I. Pak, The product replacement algorithm is polynomial, in *Proc. FOCS 2000*, IEEE Comput. Soc. Press, Los Alamitos, CA, 2000, 476–485.

[Per] E. L. Pervova, Everywhere dense subgroups of one group of tree automorphisms, *Tr. Mat. Inst. Stekl.* **231** (2000), 356–367.

[Pete] G. Pete, *Probability and Geometry on Groups*, Lecture notes for a graduate course, 2013, 203 pp.; available at `http://www.math.bme.hu/~gabor/PGG.pdf`

[Wil] J. S. Wilson, On exponential growth and uniformly exponential growth for groups, *Invent. Math.* **155** (2004), 287–303.

[Woe] W. Woess, *Random walks on infinite graphs and groups*, Cambridge U. Press, Cambridge, 2000.